

Particle–Like Description in Quintessential Cosmology

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Abstract

Assuming equation of state for quintessential matter: $p = w(z)\rho$, we analyse dynamical behaviour of the scale factor in FRW cosmologies. It is shown that its dynamics is formally equivalent to that of a classical particle under the action of 1D potential $V(a)$. It is shown that Hamiltonian method can be easily implemented to obtain a classification of all cosmological solutions in the phase space as well as in the configurational space. Examples taken from modern cosmology [14, 15] illustrate the effectiveness of the presented approach. Advantages of representing dynamics as a 1D Hamiltonian flow, in the analysis of acceleration and horizon problems, are presented. The distant supernovae type Ia data are used to reconstruct the expansion scenario. The inverse problem of reconstructing the Hamiltonian dynamics (i.e. potential function) from the luminosity distance function $d_L(z)$ for supernovae is also considered.

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I. INTRODUCTION

The Newtonian analogue to the Friedmann-Robertson-Walker models (hereafter FRW) was first considered by Milne and McCrea who used classical mechanics concepts to describe the expansion of the universe [1, 2]. This approach has some difficulties connected with description of an infinite homogenous Newtonian cosmology [4]. Pressure, in contrast to Newtonian theory, has only relativistic character [5]. In this way, a quintessential universe has no Newtonian analogues. However, pressure seems to be play important role in the evolution of the real universe. Recently found accelerated expansion of our universe is due to the presence of a certain vacuum energy with the negative pressure equation of state [6]-[8]. This type of energy is called dark energy and equation of state, in the general quintessential form, is usually postulated in the analysis of dynamics [9].

Lima et al. [10] considered the problem of reduction of FRW cosmologies with the equation of state $p \propto \rho$ to the problem of classical particle under 1-dimensional homogenous potential. We generalize this description for the FRW quintessential cosmologies.

Recent observations of supernovae (SNIa) have revealed that this dark energy dominates the evolution of the universe. The most natural candidate to represent dark energy seems to be the cosmological constant. However, it is necessary to introduce a fine tuning of 120 orders of magnitude in order to obtain the agreement with observations. Another popular idea is the so-called quintessence, a self-interacting scalar field [11], but the quintessence program suffers from the fine tuning of microphysical parameters. In this work, we discuss the possibility that the dark energy is characterized by the equation of state

$$p = w(a)\rho + 0 = w(a(z))\rho + 0, \quad (1)$$

where $z+1 = a^{-1}$ is the relation between redshift z and the scale factor $a(t)$ in FRW models.

The main aim of this paper is to reduce the FRW dynamics (with the equation of state in the form (1)) to the form of the Newtonian equation of motion for a unit mass particle in 1D potential. Therefore, we extend classical results [1]-[4] to the quintessential cosmologies. On the other hand, the question concerning the reconstruction of the equation of motion (the same concerns reconstruction form of equation of state [11]) from the observational data becomes natural. We consider problem of reconstructing the potential function for the particle-like description of dynamics.

Organization of our paper is the following. In section 2, different methods of reducing the FRW quintessential cosmology (Q.C.) to the 1D Hamiltonian flow is presented. Some applications of this formalism are given in section 3. In section 2, we present simple analysis methods of classical cosmological problems such as the horizon problem, acceleration or cosmological constant problems. Finally, in section 4 the problem of reconstructing dynamics (potential function) from $d_L(z)$ in a model independent manner (as far as possible) is investigated.

II. HAMILTONIAN DYNAMICS OF THE Q.C. MODELS

We consider dark energy description in terms of single $w(a(z))$ but it is always possible to treat the equation of state of dark energy in terms of total pressure versus total density ratio. Moreover, in any case our treatment can be generalized to the case of many-component of non-interacting fluids.

After assuming the quintessential fluid with the equation of state (1), where p is the pressure and ρ the energy density, the dynamics of homogenous and isotropic models can be described by the following set of equations

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(1 + 3w(a))\rho + \frac{\Lambda}{3}, \quad (2)$$

$$\frac{d\rho}{dt} = -3\frac{\dot{a}}{a}(1 + w(a))\rho. \quad (3)$$

Equation (2) is the Raychaudhuri equation while eq. (3) is the continuity equation for fluid with density ρ and equation of state (1), $\dot{} \equiv \frac{d}{dt}$, t – cosmological time.

The first integral of system (2)-(3) is the Friedmann equation

$$\rho = 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} - \Lambda, \quad (4)$$

where $H = d(\ln a)/dt$ is the Hubble function, Λ is the cosmological constant and k is the curvature index.

After substitution (4) into (2) we obtain

$$\ddot{a} + \psi(a)\dot{a}^2 + \kappa(a) = 0, \quad (5)$$

where

$$\psi(a) \equiv \frac{1 + 3w(a)}{2a}, \quad (6)$$

$$\kappa(a) \equiv \frac{1}{2} \left[(1 + 3w(a)) \frac{k}{a} - (1 + w(a)) \Lambda a \right]. \quad (7)$$

There are two different methods of reducing (5) to the form of the Newtonian equation of motion (i.e. of elimination of the $\psi(a)\dot{a}^2$ type term from (5)).

1. The reparametrization of the time variable

$$t \rightarrow \tau : \quad dt = \varphi(a(\tau)) d\tau, \quad (8)$$

$$\varphi \equiv \exp \int^a \psi(a) da \quad (9)$$

reduces basic equation (5) to

$$a'' \equiv \frac{d^2 a}{d\tau^2} = -\frac{\partial V}{\partial a}, \quad (10)$$

where

$$V(a) = \int^a \kappa(a) \varphi^2(a) da \quad (11)$$

is the potential function for system (10). Finally, we obtain the dynamics reduced to the Hamiltonian flow in the 1-dimensional configurational space.

$$\mathcal{H}(a, \dot{a}) = \frac{\dot{a}^2}{2} + V(a). \quad (12)$$

It can be checked that $\mathcal{H} = E = \text{const.}$ is a constant of motion provided that energy density ρ satisfies continuity condition (3). Therefore, trajectories of the system in the phase space (a, \dot{a}) lie on the energy level $\mathcal{H} = E = \text{const.}$ (in the case of the vacuum cosmology we have $E = 0$). Of course, the Hamiltonian constraint should be consistent with the form of first integral (4). Hence, from (4) we obtain

$$\frac{\dot{a}^2}{2} + V(a) = \frac{\dot{a}^2}{2} + \frac{k}{2} - \frac{\rho a^2}{6} - \frac{\Lambda a^2}{6},$$

i.e. the potential for the system (12) is

$$2V(a) = \varphi^2(a) \left(k - \frac{\rho}{3} a^2 - \frac{\Lambda a^2}{3} \right) \equiv -\varphi^2(a) \frac{\rho_{\text{eff}}}{3} a^2. \quad (13)$$

Now, the physical trajectories lie on the zero-energy level $\mathcal{H} = E = 0$ which coincides with the form of the first integral, because now the curvature and Λ -terms are included into the effective density ρ_{eff} . Formally, the curvature term can be absorbed into the potential

function by postulating the curvature fluid for which $w_k = -1/3$, $\rho_k = -3k/a^2$ (as well as cosmological constant term, where $p_\Lambda = -\rho_\Lambda$, $\rho_\Lambda = \Lambda$).

From (12) and (13) we can see that both Hamiltonians

$$\mathcal{H} = \frac{1}{2} \left(\frac{da}{dt} \right)^2 + V(q), \quad \bar{\mathcal{H}} = \frac{1}{2} \left(\frac{da}{d\tau} \right)^2 + \varphi^2(a) V(q)$$

reproduce, in the consistent way the same Friedmann equations and they also reproduce equivalent equation of motion, because the gauge freedom in choosing the lapse function (i.e. freedom in reparametrization of time $t \rightarrow \tau$: $dt = \varphi d\tau$: $\bar{\mathcal{H}} = \varphi^2(a) \mathcal{H}$).

Let us consider special case of constant w . We have

$$\begin{aligned} \psi(a) &= \frac{1+3w}{2} a^{-1}, & \varphi &= a^{\frac{1+3w}{2}}, \\ V(a) &= \left(\frac{1}{2}k - \frac{\Lambda}{6}a^2 + \frac{\rho_0}{a^{1+3w}} \right) a^{1+3w}, \\ \mathcal{H} \rightarrow \bar{\mathcal{H}} &= a^{1+3w} \left(\frac{1}{2}\dot{a}^2 + V(a) \right). \end{aligned}$$

Classical equations of motion are

$$\dot{a} = \frac{\partial}{\partial \dot{a}}(N\mathcal{H}), \quad \ddot{a} = -\frac{\partial}{\partial a}(N\mathcal{H}), \quad (14)$$

where

$$N = a^{1+3w}.$$

The lapse function N plays the role of a Lagrange multiplier and upon its variation we obtain Hamiltonian constraint $\mathcal{H} = 0$.

The cases of constant quintessential coefficient w are important in applications. The forms of potential function for different kinds of matter and flat model are presented in Table I. Of course the presented formalism can be simply generalized to the case of any mixture of noninteracting multifluids with the equation of state $p_i = w_i \rho_i$ [13]. For our purpose it is useful to put the dynamical equations into a new form by using dimensionless quantities

$$x \equiv \frac{a}{a_0}, \quad T \equiv |H_0|t, \quad \Omega_{i,0} = \frac{\rho_{i,0}}{\rho_{\text{cr},0}} \quad (15)$$

with

$$H = \frac{\dot{a}}{a}, \quad \rho_{\text{cr},0} = 3H_0^2.$$

TABLE I: The forms of potential function for different kinds of matter and flat model

	stiff mat.	rad.	dust	string	top. def.	Λ	phantom
w	1	1/3	0	-1/3	-2/3	-1	-4/3
V	$\propto a^{-4}$	$\propto a^{-2}$	$\propto a^{-1}$	const.	$\propto a$	$\propto a^2$	$\propto a^3$

The subscript 0 means here that the quantity with this subscript is evaluated today (at time t_0). Additionally, we define $\Omega_k = -3k/6H_0^2$ and $\Omega_\Lambda = \Lambda/3H_0^2$. Then the Hamiltonian is

$$\mathcal{H} = \frac{\dot{x}^2}{2} + V(x), \quad (16)$$

where

$$V(x) = -\frac{1}{2}\Omega_{k,0} - \frac{1}{2} \sum_i \Omega_{i,0} x^{1-3w_i} \quad (17)$$

which should be considered on the zero-energy level.

The basic dynamical equations are then rewritten as

$$\begin{aligned} \frac{\dot{x}^2}{2} + V(x) &= 0, \\ \ddot{x} &= \frac{1}{2} \sum_i \Omega_{i,0} (1 - 3w_i) x^{-3w_i}. \end{aligned} \quad (18)$$

In the general case, potential function can be obtained from ρ_{eff} i.e.

$$V(a) = -\frac{\rho_{\text{eff}}(a)}{6} a^2, \quad (19)$$

where for the quintessence matter we have

$$\begin{aligned} \rho_{\text{eff}} &= \rho_\Lambda + \rho_k + \rho_0 a^{-3} \exp \left\{ -3 \int^a \frac{w(a)}{a} da \right\} \\ \rho_{\text{eff}} &= \Lambda - 3k(1+z)^2 \\ &\quad + \rho_0(1+z)^3 \exp \left\{ 3 \int^a \frac{w(a(z))}{1+z} dz \right\}. \end{aligned} \quad (20)$$

2. In the second approach to the FRW dynamics representation by the Newtonian equation of motion, we redefine the position variable a , namely we define X in the following way

$$a \rightarrow X \equiv a^{D(a)}, \quad (21)$$

where $D(a)$ is chosen in such a way that the term with \dot{a}^2 is absent in (5). Hence we obtain new variable

$$X = \int^a \varphi(a) da, \quad D(a) = \log_a \int^a \varphi(a) da \quad (22)$$

and the equation of motion takes the form

$$\ddot{X} = -\kappa(a(X))\varphi(a(X)) = -\frac{\partial V}{\partial X}. \quad (23)$$

In the special case of constant w we have

$$D(w) = \log_a \int^a a^{\frac{1+3w}{2}} da = \log_a a^{\frac{3}{2}(1+w)} = \frac{3}{2}(1+w),$$

and because of the obvious formula

$$V(a(X)) = \int^a \kappa(a)\varphi(a) dX, \quad (24)$$

in the considered case, we obtain

$$\varphi = a^{\frac{1+3w}{2}}, \quad X = a^{\frac{3}{2}(1+w)}. \quad (25)$$

$$\mathcal{H}(X, \dot{X}) = \frac{\dot{X}^2}{2} + V(X) \equiv 0,$$

$$\begin{aligned} V(X) &= \frac{3}{2}(1+w) \left\{ \frac{k}{2} - \frac{\Lambda}{6} X^{\frac{2}{D}} + \frac{\rho_0}{X^{\frac{1+3w}{D}}} \right\} X^{\frac{1}{D}(1+3w)} \\ &= \frac{3}{2}(1+w) \left\{ \frac{k}{2} X^{\frac{2(D-1)}{D}} - \frac{\Lambda}{6} X^2 + \rho_0 \right\}. \end{aligned} \quad (26)$$

Let us observe that due to the new variable X , the considered system assumes the very simple form. It can be now considered on the energy level $\mathcal{H} = E > 0$, $E = \frac{3}{2}(1+w)\rho_0$ and $V(X) = \frac{k}{2}X^{2(1-\frac{1}{D})} - \frac{\Lambda}{6}X^2$. This system was analyzed on the phase plane (X, \dot{X}) many years ago from the point of its structural stability. It is interesting that there exists its natural generalization to the case of quintessential cosmology.

Let us now emphasize some advantages of the considered approach. In general, the choice of

$$X \equiv \int^a \sqrt{a} \exp \left(\frac{3}{2} \int^a \frac{w(a')}{a'} da' \right) da \quad (27)$$

makes it possible to reduce Q.C. to the nonlinear particle mechanics. There are many advantages of using language of nonlinear mechanics. Let us consider some of them.

1. Representation of dynamics as a one-dimensional Hamiltonian flow allows us to make the classification of possible evolution paths in the phase space as well as in the configuration space. Then the discussion of the existence and stability of critical points can be performed based on the geometry of the potential function. It is so because the stability of critical points is determined by the Hessian $(\partial^2 \mathcal{H} / \partial x^i \partial y^i)$. In our case the Hamiltonian function assumes the very simple form characteristic for simple mechanical systems for which the Lagrangian function is natural, i.e. quadratic in velocities $\mathcal{L} = \frac{1}{2} g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta - V(q)$. In our case, $g_{\alpha\beta} = \text{const.}$ Then the characteristic equation for linearization of the system is $\lambda^2 + \det A = 0$, where λ_i are eigenvalues of the linearization matrix A of the Hamiltonian system. Therefore, the only possible critical points in a finite domain of the phase space are centres ($\det A > 0$) or saddles ($\det A < 0$). Of course, we can explore dynamics given by the canonical equations. Denoting $x = a$, $y = \dot{a} = \dot{x}$ we obtain 2D dynamical system in the form

$$\begin{aligned} \dot{x} &= \frac{\partial \mathcal{H}}{\partial y} = y, \\ \dot{y} &= -\frac{\partial \mathcal{H}}{\partial x} = -\frac{\partial V}{\partial x}. \end{aligned} \quad (28)$$

We can observe that trajectories are integrable in quadratures. Namely, from the Hamiltonian constraint $\mathcal{H} = E = 0$ we obtain the integral

$$t - t_0 = \int_{a_0}^a \frac{da}{\sqrt{-2V(a)}}, \quad (29)$$

and for some specific forms of the potential function we can obtain the exact solutions.

2. It is possible to make the classification of qualitative evolution paths by analyzing the characteristic curve which represents the boundary equation in the configuration space. For this purpose we consider the equation of the zero velocity, $\dot{a} = 0$, which represents the boundary of the domain admissible for motion $\mathcal{D}_E = \{a \in \mathbb{R}_+ : 2(E - V) \geq 0\}$. By considering the boundary of \mathcal{D}_E given by the condition

$$\partial \mathcal{D}_E = \{a : V(a) = 0\},$$

full qualitative classification of evolutionary paths can be performed.

3. We can find the domains of cosmic acceleration as well as the domains for which the horizon problem is solved. Because $\ddot{a} = -dV/da$, one can easily see that acceleration of the universe takes place if $V(a)$ is a decreasing function of its argument. The condition $\partial V / \partial a|_{a=a_0}$ determines the static critical point on the phase plane.

Another interesting question concerns the horizon problem. It is easy to prove the following criterion of avoiding this problem. The FRW cosmological model does not have an event horizon near the singularity if $\dot{a}(t)c^{-1}$ tends to a constant while $a(t)$ tends to zero [12].

When all events whose coordinates in the past are located beyond some distance d_H then can never communicate with the observer at the coordinate $r = 0$ (in R-W metric). We can define the distance d_H as the past event horizon distance

$$d_H(t) = a(t) \int_{t_0}^t \frac{dt'}{a(t')} c = a(t) I.$$

Of course, whenever I diverges as $t_0 \rightarrow 0$, there is no past event horizon in the spacetime geometry [12]. Then it is in principle possible to receive signals from sufficiently early universe from any comoving particle like a typical galaxy. If the t' integral converges for $t_0 \rightarrow 0$ then our communication with observer at $r = 0$ is limited by what Rindler has called a particle horizon. The particle horizon will be present if energy density is growing faster than $a^{-2-\epsilon}$ as $a \rightarrow 0$ ($\epsilon \geq 0$) [12]. Therefore if $\dot{a} > a^{-\epsilon/2}$ ($\dot{a} > A$) then there is a particle horizon in the past.

Let $c^{-1}\dot{a} < A$ ($c = \text{const.}$ is assumed but the corresponding theorem can be established for the case of variable $c(t)$ [13]). Then $I \geq \frac{1}{A} \int_0^{a_0} \frac{da}{a} = \frac{1}{A}(\ln a_0 + \infty)$. Therefore, I diverges as $a \rightarrow 0$ and there exists no past horizon if the velocity of expansion factor $\dot{a} \leq A$ is bounded. Our investigation of the particle horizon is independent of any specific assumption about the behaviour of $a(t)$ near the singularity or of specific form of the equation of state. If one assumes a linear equation of state $p = w\rho$ and $w = \text{const.}$, then Friedmann's equations imply the following behaviour for $a(t) \simeq (t)^{\frac{2}{3(\gamma+1)}}$ near the singularity $t = 0$. The integral I would thus diverge only if $\gamma < -1/3$, i.e. only if the pressure p becomes negative. This is the condition for solving the horizon problem and it is identical to that for the solution of the flatness problem ($8\pi G\rho/3$ term will dominate the curvature term in a long time evolution). We can also show here that the integral $\int \frac{dt}{a(t)}$ would diverge only if the pressure of the cosmological fluid takes negative values in the general case of $w(a(t))$. Conservation condition can be rewritten in the form

$$\frac{dp}{dt} = \frac{d}{dt}[\rho a^3(1 + w(a))].$$

We can verify that boundedness of $\dot{a}(t)$ means also that ρa^2 remains bounded near the singularity (see FRW eq.). Therefore, $\rho a^3 \rightarrow 0$ as $t \rightarrow 0$. By integrating both sides of the equation written above from 0 to t we obtain

$$-3 \int_0^t p a^2 \dot{a} dt = a^3 \rho(t) \geq 0.$$

Consequently, $w(a)$ must assume negative value without any specific assumption about the equation of state.

The above criterion can be now formulated in the language of the phase space variables. If $V(a) \leq \text{const.}$ (it does not diverge at singularity) as $a \rightarrow 0$ then there exists no past horizon of the particle.

4. Another natural generalization of the presented formalism consists in including anisotropy in simple Bianchi models (Bianchi I or Bianchi V models) or VSL models with variable velocity of light of the Albrecht–Maguejo–Barrow type.

In the case of model BI ($k = 0$) or BV ($k = -1$), basic equation assumes the generalized form

$$\frac{\ddot{a}}{a} = -\frac{2}{3}\sigma^2 - \frac{1}{6}(1 + 3w(a))\rho + \frac{\Lambda c^2}{3}, \quad (30)$$

where $c(a) = c_0 a^n$ (in the AMB parametrization).

Equation (30) has generalized first integral

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{kc^2}{a^2} + \frac{\sigma^2}{3} + \frac{\Lambda c^2}{3}, \quad (31)$$

where $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$ is the shear scalar which can be used to reduce dynamical equations to form (5), but now $\psi(a)$ preserves its form whereas $\kappa(a) \rightarrow \bar{\kappa}(a)$ takes the new one which builds the new potential function.

$$\begin{aligned} \bar{\kappa}(a) &= \frac{c^2(a)}{2} \left[(1 + 3w(a)) \frac{k}{a} - (1 + w(a)) \Lambda a \right] \\ &\quad + \frac{\sigma^2 a}{2} (1 - w(a)), \\ V(a) &\rightarrow \bar{V}(a) = \int^a \bar{\kappa}(a) \varphi^2(a) da. \end{aligned} \quad (32)$$

5. Let us now consider a class of FRW cosmologies with dissipation in the form of bulk viscosity. Owing to the assumed spacetime FRW symmetries the shear viscosity vanishes,

and we deal only with bulk viscosity dissipation. It was given by Weinberg who classified the physical significance of bulk viscosity effects within the framework of general relativity.

The presence of bulk viscosity is equivalent to introducing effective pressure

$$p_{\text{eff}} = w(a(z))\rho - 3\xi H, \quad (33)$$

where ξ is the bulk viscosity coefficient (constant for the sake of simplicity).

If we introduce new variable X following previously described scheme, we obtain the generalization of (23) in the form

$$\ddot{X} - \alpha \dot{X} + \frac{\partial V}{\partial X} = 0. \quad (34)$$

Equation (34) has the form of nonlinear oscillator equation with the damping force $\propto \dot{X}$ and the exciting force $F(X) = -\frac{\partial V}{\partial X}$. In the following, this system will be analyzed qualitatively by using methods of dynamical system. Therefore, on the basis of the corresponding reduction, the particle-like description may be given if effects of bulk viscosity are included.

6. If we specialize the standard situation in cosmology with $\Lambda = 0$ and fluid which satisfy weak energy condition $p + \rho \geq 0$ in which we set $p = w\rho$ and $w = \text{const.}$ then the conservation equation gives $\rho \propto a^{-3(1+w)}$ and the $\rho/3$ term will dominate the curvature term ka^{-2} at large a so long as the matter stress violate strong energy condition $\rho + 3p < 0$ and obey $p + \rho \geq 0$, that is if $-1 \leq w \leq -1/3$. This is what we shall mean by the flatness problem. The scale factor then evolves as $a(t) \propto t^{\frac{2}{3(w+1)}}$ if $w > -1$ or $\exp(H_0 t)$, H_0 constant if $w = -1$.

In the general case of variable $w(a(z))$ quintessential model will provide a solution of flatness problem if zero curvature solution is representing late time attractor in the phase plane. This is possible in terms of potential function if $\dot{a} > \text{const.}$ as $a \gg 1$ or

$$-C^2/2 < V < 0 \quad (35)$$

This is the condition opposite to that for the solution of horizon problem $V < -C^2/2$ but it is formulated rather for large a than for small a . Let us notice that if $\ddot{a} > 0$ as $a \gg 1$ then $\dot{a} > Ct$ and period of accelerated expansion in long term behaviour can solve the flatness problem.

Let us consider the case of non-zero cosmological constant added to the potential function. Then in order to explain why it does not totally dominate matter term $\rho/3$ at large a we would need the presence of fluid with subnegative pressure, i.e. with $w < -1$ such that a

weak energy condition is violated $p + \rho < 0$. This is what we shall mean by the cosmological constant problem. If we assume that ρ is decreasing function of scale factor the solution of cosmological constant problem leads to contradiction $\dot{a} < 0$. In the general case we have following condition for solving Λ -problem

$$\rho_{\text{eff}} > \Lambda \Leftrightarrow \frac{d}{da} \left(\frac{V}{a^2} \right) > 0 \Leftrightarrow |V| > \Lambda a^{-2} \quad (36)$$

i.e. modulus of potential function is growing faster than a^{-2} at large a .

Let us notice that for $w = \text{const.}$ above condition leads to the identical condition to that to solve the horizon problem $-1 \leq w \leq -1/3$. It is consequence of the fact that corresponding conditions are valid at different domains $a \rightarrow \infty$ and $a \rightarrow 0$.

III. APPLICATIONS

We assume now that quintessential coefficient $w(a(z)) \equiv p/\rho$ is the analytical function of z

$$w(a(z)) = w(0) + \frac{dw}{dz} \Big|_0 z + \frac{1}{2} \frac{d^2 w}{dz^2} \Big|_0 z^2 + \dots \quad (37)$$

Let $\gamma_0 = w(0)$ and

$$\gamma_1 = \frac{dw}{dz} \Big|_0, \quad \gamma_2 = \frac{1}{2} \frac{d^2 w}{dz^2} \Big|_0, \quad \dots, \quad \gamma_i = \frac{1}{i!} \frac{d^i w}{dz^i} \Big|_0.$$

We consider models with vanishing Λ -term because conception of quintessence is treated as an alternative to the cosmological constant. The potential function of the dynamical system describing the FRW model with equation of state (37) is given by

$$\begin{aligned} V &= \int^a \kappa(a) \varphi^2(a) da = \frac{k}{2} a \exp \left\{ 3 \int^a \frac{w(a)}{a} da \right\} \\ &= \frac{k}{2} (1+z)^{-1} \exp \left\{ (-3) \int^z \frac{w(a(z))}{(1+z)} dz \right\}. \end{aligned} \quad (38)$$

For example, in the case of mixture of noninteracting dust and fluid for which $p_x = w_x \rho_x$, where $w_x = \text{const.}$, we have

$$V(a) = \frac{k}{2} a \left(\frac{a^{3w_x}}{\frac{\rho_{0m}}{\rho_{0x}} a^{3w_x} + 1} \right), \quad (39)$$

here we consider quintessential matter for which

$$\rho_x = \rho_{0x} a^{-3(1+w_x)}, \quad \rho_{0x} = \text{const.}$$

and dust matter for which

$$\rho_m = \rho_{0m} a^{-3}, \quad \rho_{0m} = \text{const.}$$

If for quintessential fluid $w_x = w_x(a)$, then the potential function takes the more general form

$$V(a) = \frac{k}{2} a \left[\frac{\exp \left(3 \int^a \frac{w_x(a)}{a} da \right)}{\frac{\rho_{0m}}{\rho_{0x}} \exp \left(3 \int^a \frac{w_x(a)}{a} da \right) + 1} \right]. \quad (40)$$

After substituting (37) into (38) we obtain the general form of the potential function which can be useful for a classification of cosmological models in the configurational space

$$V(a(z)) = \frac{k}{2} (1+z)^{-(1+3\gamma_0)} \times \exp \left\{ (-3) \int^z \frac{(\gamma_1 z + \gamma_2 z^2 + \dots)}{(1+z)} dz \right\}. \quad (41)$$

Let us consider a few special cases of (41).

A.

Let $\gamma_0 = -\frac{1}{3}$ (like for a string) which determines the boundary the strong energy condition violation. Additionally, we consider expansion of $w(z)$ up to the second order term. Then we obtain from the integral

$$\int \frac{\gamma_1 z + \gamma_2 z^2}{1+z} dz = (\gamma_1 - \gamma_2)z + \frac{\gamma_2}{2} z^2 - (\gamma_1 - \gamma_2) \ln(1+z) \quad (42)$$

the exact form of the potential function

$$V(a(z)) = \frac{k}{2} e^{-3(\gamma_1 - \gamma_2)z - \frac{3}{2}\gamma_2 z^2} (1+z)^{3(\gamma_1 - \gamma_2)}, \quad (43)$$

or for any γ_0

$$V(a(z)) = \frac{k}{2} e^{-3(\gamma_1 - \gamma_2)z - \frac{3}{2}\gamma_2 z^2} (1+z)^{-(1+3\gamma_0)+3(\gamma_1 - \gamma_2)}. \quad (44)$$

Of course, the trajectories of the considered system lie in the domain admissible for motion defined as

$$\mathcal{D}_E = \{a \in \mathbb{R}_+ : 2(E - V(a)) \geq 0, E = \frac{\rho_0}{6}\},$$

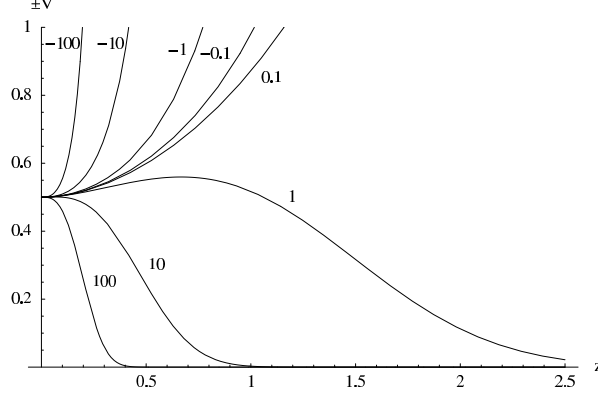


FIG. 1: Diagram of $V(z)$ (formula (44)) for the case of $k = \pm 1$, $\gamma_0 = -1/3$, $\gamma_1 = -2/3$ and various γ_2 ($\gamma_2 = -100, -10, -1, -0.1, 0.1, 1, 10, 100$)

$$\rho = \rho_0 a^{-3} \exp \left(3 \int \frac{w(a)}{a} da \right).$$

Therefore, equation of the boundary $\partial \mathcal{D}_E$ is

$$V = E \Leftrightarrow \frac{k}{2} e^{-3(\gamma_1 - \gamma_2)z - \frac{3}{2}\gamma_2 z^2} = E(1+z)^{(1+3\gamma_0)-3(\gamma_1-\gamma_2)}.$$

If we substitute $\gamma_0 = -\frac{1}{3}$, then we obtain the potential function expressed in terms of a single parameter $\gamma_1/\gamma_2 \equiv x$, namely

$$\left(\frac{1+z}{e^z} \right)^{(x-1)} = \left(\frac{2E}{k} \right)^{\frac{1}{3\gamma_2}} e^{z^2/2} = \bar{E} e^{z^2/2}.$$

Finally, we obtain

$$x(z) = 1 + \frac{\ln \bar{E} + z^2/2}{\ln(1+z) - z}, \quad (45)$$

where $\bar{E} = (2E/k)^{1/3\gamma_2}$.

The plot of $x(z)$ for different \bar{E} is shown in Fig. 2.

Let us note that if $\rho_0 > 0$ ($E > 0$), then there is no boundary for $k = -1$, whereas if $\rho_0 < 0$ ($E < 0$), there is no boundary for $k = +1$ of the domain admissible for motion. Finally, we consider the evolution path as a level of $x = \text{const.}$ and then we classify all evolutions modulo the quantitative properties of their dynamics [22].

B.

$\gamma_2 = 0$, $\gamma_0 \neq 0$, i.e. we consider quintessential model with pressure $p = \gamma_0 \rho + (\gamma_1 z) \rho$.

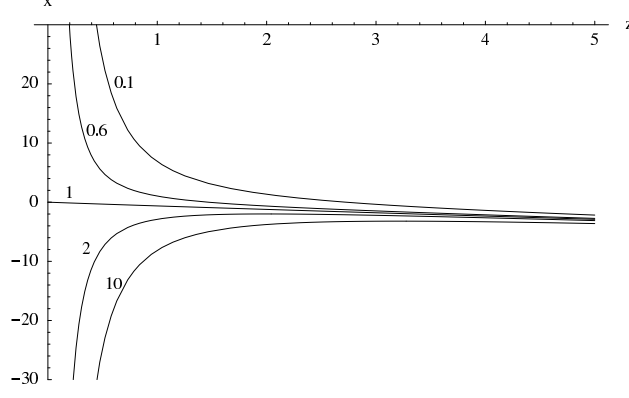


FIG. 2: Diagram of dependence $x(z)$ for classification of models with potential (44) ($\bar{E} = 0.1, 0.6, 1, 2, 10$)

Then we obtain the potential in the form (Fig.3)

$$V(a(z)) = \frac{k}{2} e^{-3\gamma_1 z} (1+z)^{-(1+3\gamma_0)+3\gamma_1}. \quad (46)$$

Let us consider the boundary of the configuration space given by $V(z)$. Then γ_0 can be

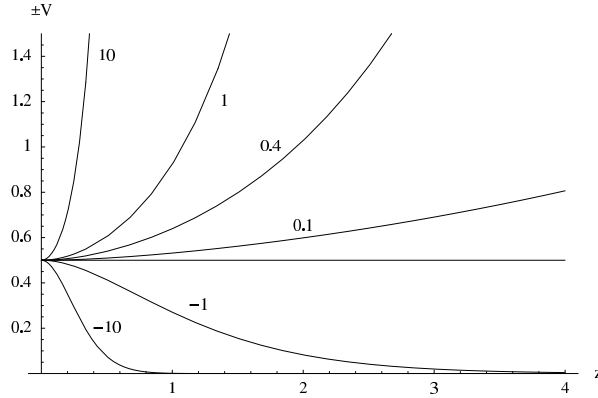


FIG. 3: Diagram of the potential function $V(z)$ for the case \mathbf{B} , $p = (\gamma_0 + \gamma_1 z)\rho$, $\gamma_0 = -1/3$, $\gamma_1 = -2/3\beta$ ($\beta = -10, -1, 0.1, 0.4, 1, 10$), $k = \pm 1$

expressed as a function of z in the following way

$$\gamma_0(z) = \left(-\frac{1}{3} + \gamma_1\right) - \frac{\gamma_1 z}{\ln(1+z)} + \frac{c/3}{\ln(1+z)}, \quad (47)$$

where $z = a^{-1} - 1$, $c = -\ln|\frac{2E}{k}|$. The plot of $\gamma_0(z)$ for different γ_1 is shown in Fig. 4 for $\rho_0 > 0$ and $k = +1$. If we put $\gamma_0 = -1$ (cosmological constant) then we can obtain classification of

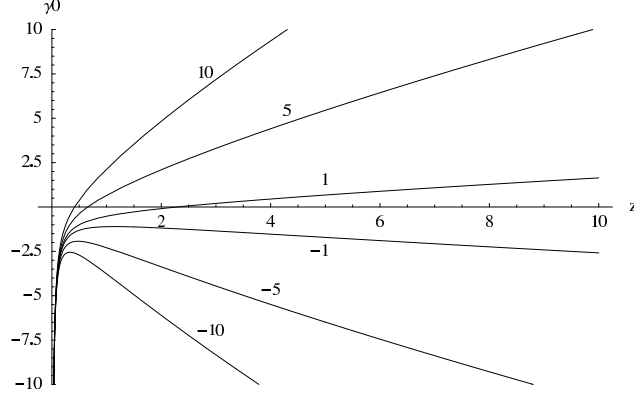


FIG. 4: Diagram of boundary curve $\partial\mathcal{D}_E$ usefull in classification of the model in the configurational space (formula (47)). We put here $\gamma_1 = -2/3\beta$, ($\beta = -10, -5, -1, 1, 5, 10$), $c = -1$

cosmological models with cosmological constant in terms of levels of $\gamma_1 = \text{const.}$, where

$$\gamma_1(z) = \frac{-\frac{2}{3} \ln(1+z) - c/3}{\ln(1+z) - z}. \quad (48)$$

The plot of $\gamma_1(z(a))$, for different c , is shown in Fig. 5.

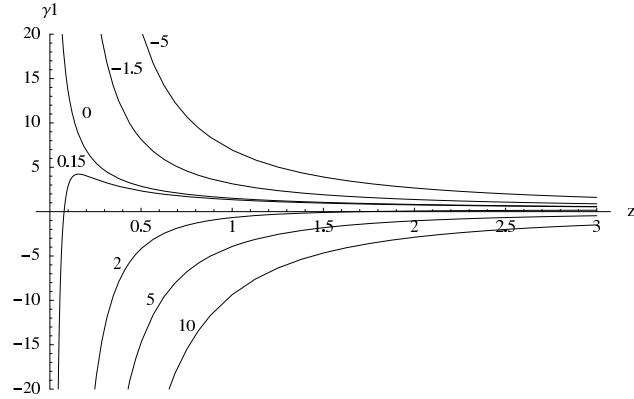


FIG. 5: The plot of $\gamma_1(z(a))$ (formula (48)) for different signs of c ($c = -5, -1.5, 0, 0.15, 2, 5, 10$)

C.

$p = w_x \rho_x$, $w_x = \text{const.}$ Let us consider quintessential matter in the form of noninteracting mixture of dust and matter described by the equation of state $p = w_x \rho_x$, where $w_x = \text{const.}$ In this case, classification of all evolutional paths can be given in terms of potential function (Fig. 6)

$$V(a) = \frac{k}{2} a \left(\frac{a^{3w_x}}{\frac{\rho_{0m}}{\rho_{0x}} a^{3w_x} + 1} \right), \quad (49)$$

where

$$\rho_x = \rho_{0x} a^{-3(1+w_x)}, \quad \rho_m = \rho_{0m} a^{-3}.$$

The boundary curve $V(a) = E$ can be used to classify all evolutionary paths in the configu-

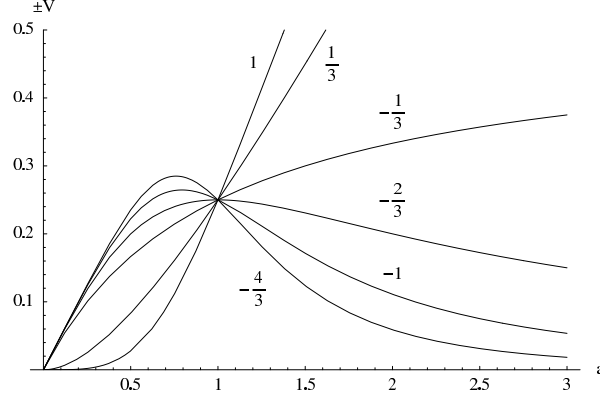


FIG. 6: Diagram of the potential function $V(a)$ for the case **C**, $p = w_x \rho_x$, for different $(w_x = -\frac{4}{3}, -1, -\frac{2}{3}, -\frac{1}{3}, 1, \frac{1}{3})$, $k = \pm 1$

rational space, namely

$$g(a) = 3w_x(a) = \frac{\ln \left(\frac{A}{a - A \frac{\rho_{0m}}{\rho_{0x}}} \right)}{\ln a}, \quad (50)$$

where $A = 2E/k$.

The plot of $g(a)$, for different A , is shown in Fig. 7.

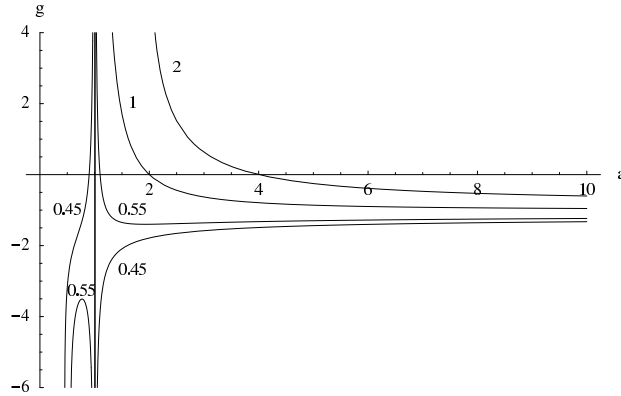


FIG. 7: Diagram of $g(a)$ for classification of evolution in the configurational space for the case of constant w_x ($A = 0.45, 0.55, 1, 2$), $\frac{\rho_{0m}}{\rho_{0x}} = 1$

D.

Since the observations of the supernovae of type Ia indicate that the universe must be today in an accelerated expansion the nature of the fluid responsible for such a behaviour has been object of many studies. While the most obvious candidate for such component is the vacuum energy the possibility that the dark energy might be described by the Chaplygin gas is seriously suggested [14].

The Chaplygin gas has an interesting motivation connected with the string theory. If we consider a d-brane configuration in the d+2 Nambu-Goto action, the employment of the light-cone parametrization leads to the action of a Newtonian fluid with the equation of state $p = -A/\rho$, whose symmetries are the same as those of the Poincare' group. Hence, the relativistic character of the action is somehow hidden in the equation of state (for review see R. Jackiv, A particle field theorist's lectures on supersymmetric, non-abelian fluid mechanics and d-branes physics).

Let us consider FRW model with fluid in the form of generalized Chaplygin gas for which equation of state is given by

$$p = -\frac{A}{\rho^\alpha}, \quad 0 \leq \alpha < 1. \quad (51)$$

The energy-momentum conservation implies that Chaplygin gas density depends on the scale factor as

$$\rho = \left(A + \frac{B}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}}, \quad (52)$$

where A and B are constants and the Chaplygin gas corresponds to the case $\alpha = 1$; A is a positive constant because sound velocity of Chaplygin gas is $v_s^2/c^2 = A^2/\rho^2$.

Equation (52) interpolates smoothly between a dust dominated phase, where $\rho \propto a^{-3}$, and a De Sitter phase, where $p = -\rho$, through an intermediate regime described by the Zeldovich stiff matter $p = \rho$.

Following our previous consideration, the dynamics is given by the hamiltonian

$$\mathcal{H} = \frac{\dot{a}^2}{2} + V(a) \equiv 0,$$

where

$$V(a) = -\frac{1}{6}\rho_{\text{eff}}a^2 = -\frac{1}{6}\left(A + \frac{B}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}} a^2 + \frac{k}{2}. \quad (53)$$

The dependence of $V(a)$ for $\alpha = 1$ is illustrated on the Fig. 8 The domain admissible for

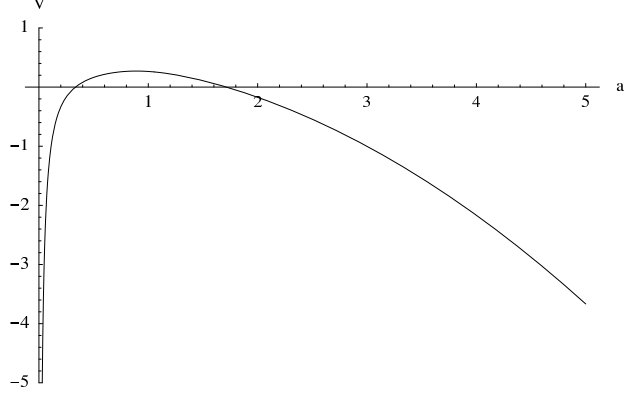


FIG. 8: Diagram of the potential function $V(a)$ for the case **D** (formula (53)), $k = A = B = \alpha = 1$.

motion of the system with generalized Chaplygin gas is

$$\mathcal{D}_0 = \{a : V(a) \leq 0\}.$$

The boundary curve $\partial\mathcal{D}_0$ can be used to classify possible evolution paths in the configurational space in the following way. We consider constant levels of $A(a)$ relation given by

$$A(a) = \left(\frac{3k}{a^2}\right)^{1+\alpha} - \frac{B}{a^{3(1+\alpha)}}. \quad (54)$$

The zero velocity curve for $k = +1$ are shown on Fig. 9 and Fig. 10.

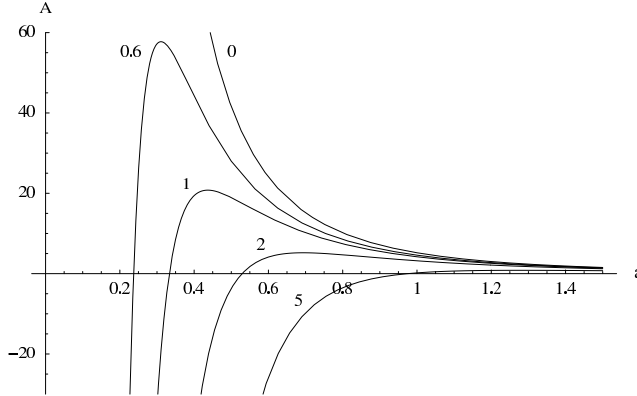


FIG. 9: Diagram of $A(a)$ for classification of possible evolution paths for the case of constant $k = +1$, $\alpha = 0.5$ and different B ($B = 0, 0.6, 1, 2, 5$)

It is interesting that the existence and character of the critical points of the considered system $\dot{a} = x$, $\dot{x} = -\frac{\partial V}{\partial a}$ depends on the geometry of the potential function.

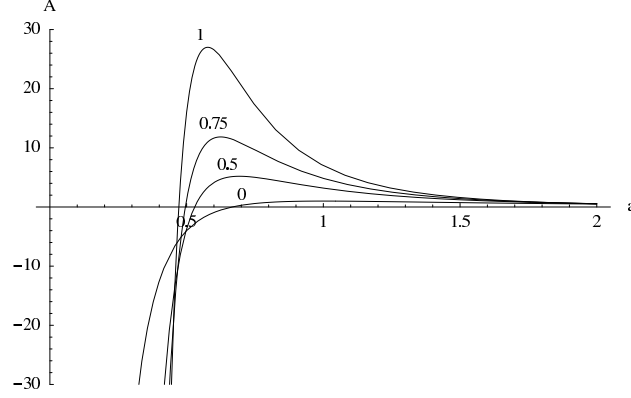


FIG. 10: Diagram of $A(a)$ for classification of possible evolution paths for the case of constant $k = +1$, $B = 2$ and different α ($\alpha = 0, 0.5, 0.75, 1$)

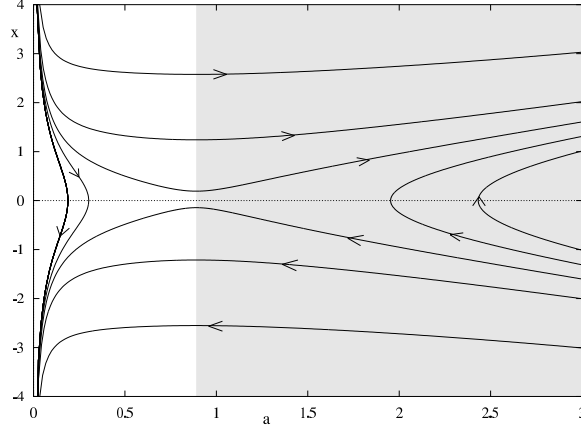


FIG. 11: The phase portrait (a, x) for case **D** ($k = A = B = \alpha = 1$). The shaded region is a region of accelerated expansion of the universe.

It can be easily shown that in our case on a finite region of the phase plane (a, x) only saddle points are admitted because $\partial^2 V / \partial a^2 < 0$ at the critical point $a = a_0$, $\partial V / \partial a|_{a=a_0} = 0$. Therefore, all points are hyperbolic ($Tr A = 0$) and the system is structurally stable.

In terms of $V(a)$ the domains of accelerating trajectories can be easily found, namely if $\partial V / \partial a < 0$ then the system starts to accelerate. Because the diagram of $V(a)$ is upper convex, the static critical point will separate regions without acceleration from the domain in the phase space where trajectories accelerate. One can show that the critical value of a is

$$a = a_{crit} = \left(\frac{B}{2A} \right)^{\frac{1}{3(1+\alpha)}}.$$

At this point the diagram $V(a)$ has the maximum.

E.

Interesting formulas for $w(a)$ were already proposed by Corasaniti and Copeland [15]. They considered a broad class of tracking potentials for scalar fields, namely $V(\phi) \propto \phi^{-\alpha}$ – inverse power law potential (INV) [16], $V(\phi) \propto \phi^{-\alpha} \exp(\phi^2/2)$ – supergravity potential (SUGRA) [17], $V(\phi) \propto \exp(-\alpha\phi) + \exp(\beta\phi)$ (2EXP) [18], The Skordis model [19] and Copeland-Nunes-Rosati model (CNR) [20].

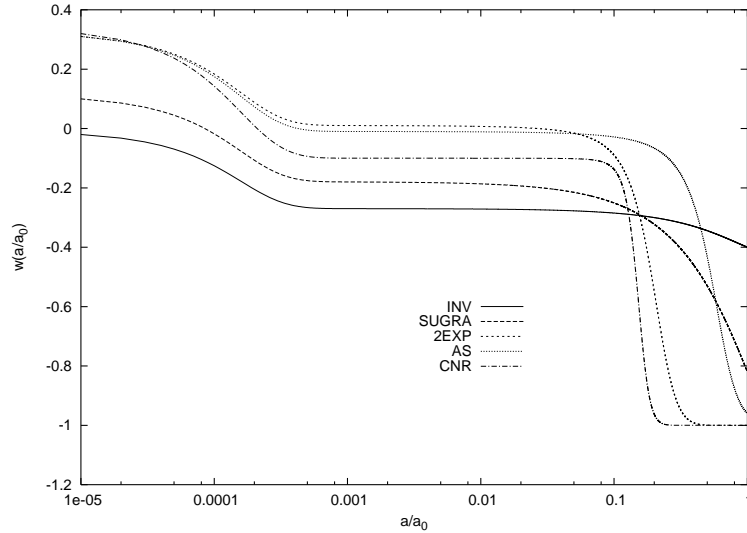


FIG. 12: The dependence of the equation of state factor $w(a)$ versus the scale factor a/a_0 for different models of potential of scalar field.

First, we apply our method to construct a potential function of the corresponding Hamiltonian dynamical system. It assumes the following form

$$\begin{aligned}
 V(a) &= -\frac{\rho_0}{6} a^{-1} \exp \left(3 \int_a^1 \frac{w(a)}{a} da \right) \\
 &= -\frac{\rho_0}{6a} \exp \left\{ 3 \left(F_1 \int_a^1 \frac{f_r(a)}{a} da + F_2 \int_a^1 \frac{f_m(a)}{a} da + F_3(1-a) \right) \right\}, \quad (55)
 \end{aligned}$$

where

$$\int_a^1 \frac{f_{r,m}}{a} da = \sum_{n=0}^{\infty} [\text{Ei}(-\beta n x)_{r,m} - \text{Ei}(1)], \quad (56)$$

where $x = a - a_c$, $\beta = 1/\Delta$, Ei – exponent integral function and coefficients F_1 , F_2 , F_3 are determined by the condition that $w(a)$ takes on the respective values of w^r , w^m and w^0 , during radiation epoch ($a = a_r$), matter domination ($a = a_m$) as well as today ($a = a_0 = 1$).

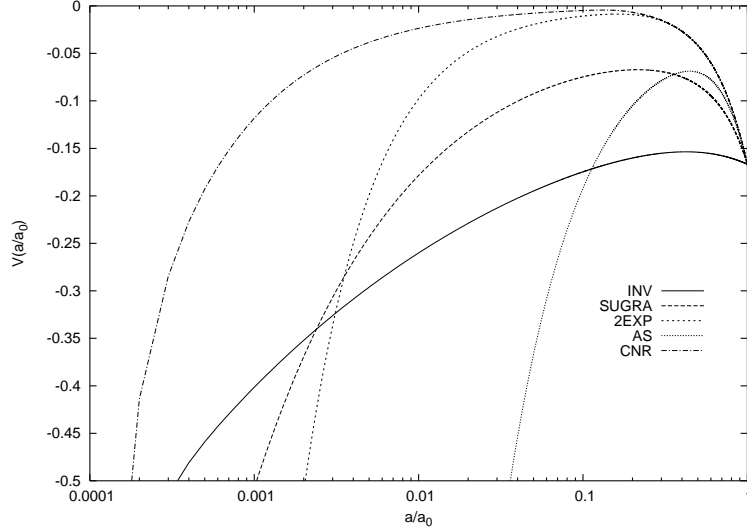


FIG. 13: The dependence of the potential of the Hamiltonian system against the scale factor a/a_0 for a different class of tracking potentials.

The corresponding forms of $w(a)$ function for a different class of potentials are illustrated in Fig. 12.

The function $f_{r,m}(a)$ has the following form [15]

$$f_{r,m}(a) = \frac{1}{1 + \exp[-(a - a_c^{r,m})/\Delta_{r,m}]}, \quad (57)$$

where the corresponding values of coefficients $a_c^{r,m}$ are taken from Table I in ref [15].

After substitution formulas (57) to (55) we obtain different forms of potentials for a different class of models (see Fig. 13). From this figure we can observe that in all cases a potential function is of the same type (upper convex), like for the Chaplygin gas. Therefore the phase portraits determined from the potential functions of the systems are topologically equivalent. From the physical point of view it means that considered models can be seriously treated as candidates for dark energy description.

Let us notice that the dynamical system

$$\dot{a} = x, \quad \dot{x} = -\frac{\partial V}{\partial a}$$

can be transformed to (z, x) variables ($a = (1 + z)^{-1}$) and then we obtain

$$\frac{dz}{d\tau} = -(1 + z)^2 x, \quad \frac{dx}{d\tau} = (1 + z)^2 \frac{\partial V}{\partial z}.$$

Therefore after the reparametrization of time along trajectories $\tau \rightarrow \eta : d\eta = -(1+z)^2 d\tau$ (now η will be a decreasing function of time variable τ) we obtain the dynamical system

$$\frac{dz}{d\eta} = x, \quad \frac{dx}{d\eta} = -\frac{\partial V}{\partial z}. \quad (58)$$

Of course, system (58) can be analysed in terms of dynamical systems, i.e. in terms of the method of qualitative analysis of differential equations on the phase plane (z, x) . System (58) has the first integral in the form

$$\left(\frac{dz}{d\eta}\right)^2 - 2(E - V(z)) = 0. \quad (59)$$

For the considered system only two types of critical points can appear, namely centres or saddle points. If $\det A = \frac{\partial^2 V}{\partial z^2} \Big|_{z=z_0}$ is negative, the diagram of the potential function $V(a(z))$ has maxima; they correspond to the saddle point. On the other hand, if the diagram of the potential function $V(a(z))$ has minima, they correspond to centres. It is important that we can discuss the stability of critical points based only on the geometry of the potential function. It is easy to check that at the critical points of the system appearing at $a_0 = \text{const.} (z_0 = \text{const.})$, $\frac{\partial V}{\partial a} \Big|_{a_0} = 0$ ($\frac{\partial V}{\partial z} \Big|_{z_0} = 0$) we have $\kappa(a) = 0$, and then

$$\begin{aligned} \frac{\partial^2 V}{\partial a^2} \Big|_{a=a_0} &= \varphi^2 \frac{3}{2} \frac{k}{a} \frac{dw}{da} \Big|_{a=a_0} = \frac{3}{2} \frac{\varphi^2}{a} k \frac{dw}{da} \Big|_{w(a)=-1/3} \\ &= -\frac{3}{2} \varphi^2 k (1+z)^3 \frac{dw}{dz} \Big|_{z=z_0}. \end{aligned} \quad (60)$$

Therefore, if $k \frac{dw}{dz}(z_0)$ is positive, only saddles points can appear which guarantee the structural stability of the system.

For the case $\mathbf{B} \frac{dw}{dz} = \gamma_1$, the above condition means that $\gamma_1 k > 0$. The phase portraits for $\gamma_0 = -1/3$, and various γ_1 , are presented on Fig. 19, 20, 21 and 22.

Let us note that critical points are located at

$$z_0 = -\left(\frac{1}{3\gamma_1} + \frac{\gamma_0}{\gamma_1}\right). \quad (61)$$

If we put, for example, $\gamma_0 = -1$ (cosmological constant term) then $\gamma_1 > -\frac{3}{2}$ critical points can only exist on a finite domain of the phase plane because $z > -1$.

The idea of structural stability comes from Andronov and Pontriagin [23]. A dynamical systems S are said to be structurally stable if their dynamical behaviour remains qualitatively (modulo homeomorphism preserving orientation of trajectories) the same (equivalent)

under small perturbations. Structural stability is sometimes considered as a precondition for the “real existence”. Structurally stable dynamical systems on the 2D compact space (for example on the plane with circle at infinity) form open and dense subsets in the space of all dynamical systems on the plane. Therefore, 2D structurally unstable models seem to be nonadequate for describing real proceses because of measurement errors.

The main aim of the qualitative analysis of differential equations is not to find, and then to analyze, individual solutions but rather to investigate space of all possible solutions for all admissible initial conditions. A property is believed to be “realistic” if it can be attributed to large (rather typical than exceptional) subsets of models within the space of all possible solutions, or if it possesses a certain stability, i.e. if it is shared by a slightly perturbed model. There is a wide opinion among specialists that realistic models should be structurally stable, or even stronger, that everything that exists should possess a kind of structural stability.

From the physical point of view it is interesting to answer the question: are the trajectories for which acceleration of the universe takes place, distributed in a typical or exceptional way? How are trajectories with interesting properties distributed in the phase plane? For example, the acceleration condition $\ddot{a} = -\partial V/\partial a > 0$ is satisfied if $V(a)$ is a decreasing function of a . For us it is important that one should be easily able to deduce this from the geometry of the potential function only.

In the phase space, the area of acceleration is determined by the condition that

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(1 + 3w)\rho > 0, \quad \frac{\partial V}{\partial a} + \psi(a)(a')^2 < 0. \quad (62)$$

(62) can be rewritten to the form which could be usefull in the analysis of the “probability of acceleration” which can be defined as a measure of the space of those initial conditions that lead to accelerating universes

$$-\frac{\partial}{\partial a} \ln(E - V) + \frac{1 + 3w(a)}{a} < 0,$$

or

$$(1 + z)^2 \frac{\partial}{\partial z} \ln(E - V(z)) + (1 + 3w(a(z)))(1 + z) < 0. \quad (63)$$

For example, domains of acceleration in the configurational space, for the case **D**, as well as the domain of $\ddot{a} > 0$ ($\frac{\partial V}{\partial a} < 0$) in the phase plane are presented on Fig. 11.

Let us now consider the presence or absence of the particle horizon in the past. Good news from our earlier discussion is that this property can be detected from the shape of the diagram of the potential function of the system, namely if $V(z)$ goes to a constant (zero is included) as $z \rightarrow \infty$ then we obtain a model without the horizon.

In the next section it will be demonstrated how we can answer the question about the horizon in the past on the base of $V(z)$ taken from the observations.

IV. INVERSE PROBLEM IN QUINTESSENTIAL COSMOLOGY

The presented formalism gives us a natural base to discuss the redshift magnitude relation $m(z)$ for SNIa supernovae observational data. But on the other hand, because the Hubble function is related to the luminosity distance, it is possible to determine both the quintessence parameter $w(z) = p/\rho$ and the potential of the dynamical system $V(z)$. Therefore, the equation of state as well as the whole dynamics can be reconstructed provided that the luminosity function $d_L(z)$ is known from observations. It is called the inverse problem in dynamics of quintessential cosmology.

As an example of constructing observables from the considered dynamics let us consider the luminosity–distance relation $d_L(z)$ for quintessential models.

If a light source of redshift z is located at a radial coordinate r_1 (taken from R–W metric), its luminosity distance d_L , its angular diameter distance d_A and its proper motion distance are given by

$$d_L(z) = (1+z)a_0r_1, \quad d_A = \frac{a_0r_1}{1+z}, \quad d_M = a_0r_1, \quad (64)$$

where r_1 calculated from metric gives

$$\begin{aligned} \varphi(r_1) &= \int_{\frac{a_0}{1+z}}^{a_0} \frac{da}{a\dot{a}} = \frac{1}{a_0} \int_0^z \frac{dz'}{H(z')} \\ &= \begin{cases} \sin^{-1} r_1 & \text{when } k = +1 \\ r_1 & \text{when } k = 0 \\ \sinh r_1 & \text{when } k = -1 \end{cases} \end{aligned} \quad (65)$$

Here a_0 is the present value of the radius of the universe. The above equation can also be

written in the form of a single compact equation as

$$\frac{d_L(z)}{1+z} = \frac{1}{\sqrt{\kappa}} \zeta \left(\sqrt{\kappa} \int_0^z \frac{dz'}{H(z')} \right), \quad (66)$$

where

$$\begin{aligned} \zeta(q) &= \sin q & \text{with } \kappa &= \Omega_{k,0} & \text{when } \Omega_{k,0} > 0, \\ \zeta(q) &= \sinh q & \text{with } \kappa &= -\Omega_{k,0} & \text{when } \Omega_{k,0} < 0, \\ \zeta(q) &= q & \text{with } \kappa &= 1 & \text{when } \Omega_{k,0} = 0. \end{aligned}$$

Thus for a given \mathcal{M} (absolute magnitude) and $H(z)$ equation (66) gives the predicted value of $m(z)$ (observed magnitude) at a given z .

By using the κ -corrected effective magnitudes m_i which have also been corrected for the light curve width–luminosity relation and the galactic extinction, and by using the same standard errors $\sigma_{2,i}$ and $\sigma_{m_i}^{\text{eff}}$ of the supernova with redshift z_i as used by Perlmutter et al. we compute χ^2 according to

$$\chi^2 = \sum_i \frac{[m_i^{\text{eff}} - m(z_i)]^2}{\sigma_{z_i}^2 + \sigma_{m_i}^2 \text{ eff}}. \quad (67)$$

The best fit parameters are obtained by minimizing equation (67).

Luminosity distance and angular distance depend sensitively on the present densities of various energy components and their equations of state. For this reason, the magnitude–redshift relation for distant standard candles has been proposed as a potential test for cosmological models.

In our formalism $H(z)$ can be immediately taken from the first integral of the dynamical equation and then we obtain

$$\frac{d_L(z)}{1+z} = \frac{1}{\sqrt{\kappa}} \zeta \left(\sqrt{\kappa} \int_0^z \frac{e^{-3/2 \int \frac{w(a(z'))}{1+z'} dz'} dz}{(1+z)^{3/2} \sqrt{2(E - V(z))}} \right). \quad (68)$$

Formula (68) limits the determination of the luminosity distance because it depends on quintessential parameter $w(z)$ through a multiple integral relation that smears out detailed information about $w(z)$. If $w(a(z))$ can be expanded as the Taylor series following (37) then we obtain the simplest formula without the double integration

$$\frac{d_L(z)}{1+z} = \frac{1}{\sqrt{\kappa}} \zeta \left(\sqrt{\kappa} \int_0^z \frac{\exp \left\{ -\frac{3}{2} \sum \gamma_i z'^i dz' \right\}}{\sqrt{2(1+z')^{3(1+\gamma_0)} (E - V(z'))}} \right). \quad (69)$$

Many authors [21] assume that a quite accurate luminosity distance may be obtained and then examined to answer the question of whether the equation of state of the expanding universe can be determined uniquely.

Our idea is more general. To determine the structure and evolution of an astrophysical system of the universe, the equation of state is usually necessary. By equation of state of the universe we mean the relation between the total energy density of cosmic matter and the total pressure. However, the equation of state relevant to the universe has not yet been established. Our idea is to reconstruct it from the form of the potential of the system $V(a(z))$.

Let us consider, for simplicity, a flat model for which the Hubble parameter is related to luminosity distance by the relation

$$H(z) = \left[\frac{d}{dz} \left(\frac{d_L(z)}{1+z} \right) \right]^{-1}. \quad (70)$$

Then it is possible to determine the quintessence parameter

$$w(z) = -1 - \frac{2}{3}H(1+z)\frac{d^2}{dz^2} \left(\frac{d_L(z)}{1+z} \right). \quad (71)$$

Here the term -1 is established from the condition that for stationary solution $H(z) = \text{const}$. $\frac{d^2}{dz^2}(\frac{d_L(z)}{1+z})$ will vanish but such a solution can appear on the phase plane (H, ρ) only as an intersection trajectory of the flat model and the boundary of the weak energy condition $\rho + p = 0$.

Equation (71) can be rewritten in the form

$$p = -\rho - 3\xi(z)H, \quad (72)$$

where

$$\xi(z) = \frac{2}{3}H^2(z)(1+z)\frac{d^2}{dz^2} \left(\frac{d_L(z)}{1+z} \right).$$

The dependence $p(H)$ manifests the presence of bulk viscosity effects in the model $\xi(z) = -\frac{1}{3}\frac{\partial p}{\partial H}$. Therefore, from $d_L(z)$ we obtain the reconstruction of $w(z)$ as a mixture of the cosmological constant term and bulk viscosity. However, in determining $w(z)$ there is an inherent limitation because the value $w(z)$ is poorly resolved and no usefull constraint can be obtained concerning its time variation. Of course, the value of $w(0)$ and dw/dz with

accuracy 0.1 and 0.15 respectively will be obtained from SNAP3 but for the reconstruction of $w(z)$ higher derivatives can be useful. If we measure $\frac{dw^i}{dz}(0)$, $i \geq 2$, then in principle, the reconstruction equation of the equation of state is possible, because then

$$w(z) = w(0) + \frac{dw}{dz}(0)z + \frac{1}{2} \frac{d^2w}{dz^2}(0)z^2 + \dots,$$

where $\frac{d^n w}{dz^n}(0)$ is determined from the recurrence formula, namely

$$\begin{aligned} \frac{d^n w}{dz^n}(0) &= \frac{2}{3} \frac{d^n}{dz^n} \bigg|_0 (\ln H) + \frac{2}{3} (1+z) \frac{d^{n+1}}{dz^{n+1}} \bigg|_0 (\ln H), \\ w(0) &= -1 + \frac{2}{3} \frac{d}{dz} \bigg|_0 (\ln H). \end{aligned} \tag{73}$$

Then $w(z)$ can be obtained from $H(z)$ in the following way

$$\begin{aligned} H(z) &\mapsto \ln H(z) \mapsto \forall n \frac{d^{n+1}}{dz^{n+1}} (\ln H) \\ &\mapsto \frac{d^n w}{dz^n}(0) \mapsto w(z) = \sum_{i/1}^{\infty} \frac{1}{i!} \frac{d^i w}{dz^i}(0) z^i + w(0). \end{aligned} \tag{74}$$

At present this idea can not be realized but the idea of reconstructing dynamics requires the knowledge of $V(z)$ which can be calculated from the Hamiltonian constraint

$$\begin{aligned} V(a(z)) &\equiv -\frac{\rho_{\text{eff}} a^2(z)}{6} = -\frac{1}{2} H^2(z) a^2(z) \\ &= -\frac{1}{2} \left[\frac{1}{(1+z) \frac{d}{dz} \left(\frac{d_L(z)}{1+z} \right)} \right]^2. \end{aligned} \tag{75}$$

On the other hand, the knowledge of $V(z)$ gives us information how the horizon problem can be solved. Reconstruction of the potential function from SNIa data is presented on Fig.14.

We obtain the plot of the potential function from fitting

$$\begin{aligned} \frac{d_L(z)}{1+z} &= \int_0^z \left[A_0 + A_1(1+z') + A_2(1+z')^2 + \right. \\ &\quad \left. + A_3(1+z')^3 + A_4(1+z')^4 \right]^{-\frac{1}{2}} dz' \end{aligned} \tag{76}$$

function to the SNIa observational data.

We can observe that as $z \rightarrow \infty$, $V(z)$ as calculated from (75) goes to $-\infty$. This can be treated as an empirical evidence of the presence horizon in the past. Let us note that other problems of the standard cosmology can be discussed analogously basing on information about $V(z)$ and its geometry.

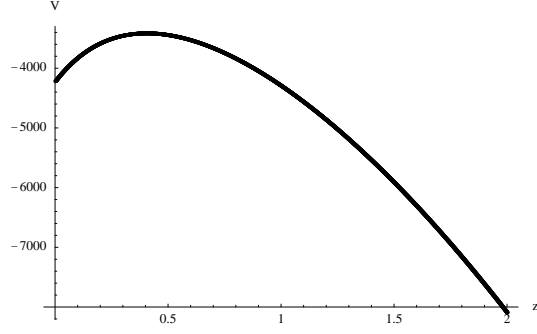


FIG. 14: The plot of the potential function $V(z)$ has been obtained from equation (75) and from fitting formula (76) to the SNIa observational data

V. CONCLUSIONS

In this work a class of FRW models with quintessence matter is examined in the context of the present acceleration of the universe. Our results are the following.

1. We have given a mathematical background for discussing physical content of quintessential cosmology. Its dynamics is reduced to the dynamics of the unit mass point particle in 1D potential. Then different physical properties, like acceleration of the universe, existence of horizon, can be formulated only in terms of the potential function of the system. The proof of the corresponding condition is quite general. In particular, it is independent of any specific assumption about the behaviour of the scale factor near the singularity (such as the assumption of power law behaviour) or a specific form of equation of state.
2. The dynamics is formulated in the hamiltonian formalism and the full classification of possible evolutions in the phase plane as well as in configurational space is given. In the near future, it will be possible to obtain, from SNAP3, the exact value of γ_0 and γ_1 appearing in the equation of state $p = (\gamma_0 + \gamma_1 z)\rho$, and then we could automatically answer the question about the horizon.
3. The effectiveness of our treatment of dynamics of quintessential models in terms of single-particle mechanics is demonstrated for a broad class of tracking potentials. We obtain topological equivalence of the phase portraits (for this case) with the dynamical system obtained for the potential function reconstructed from SNIa data.

4. The idea of reconstructing dynamics instead of quintessential coefficient is considered. It is what we called inverse problem in quintessential cosmology. We demonstrate that the reconstructed potential function of the system produces the particle horizon in the past and solves the flatness problem.

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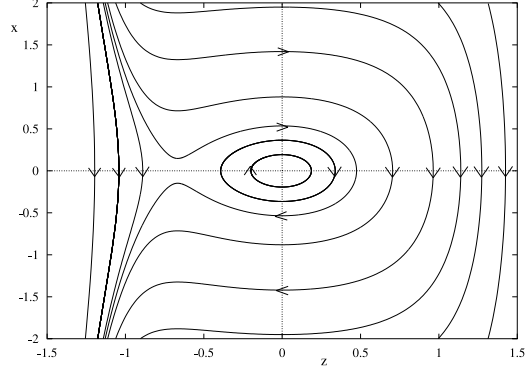


FIG. 15: The phase portrait (z, x) for case **A** $p = (\gamma_0 + \gamma_1 z + \gamma_2 z^2)\rho$ ($k = 1, \gamma_2 = -1$)

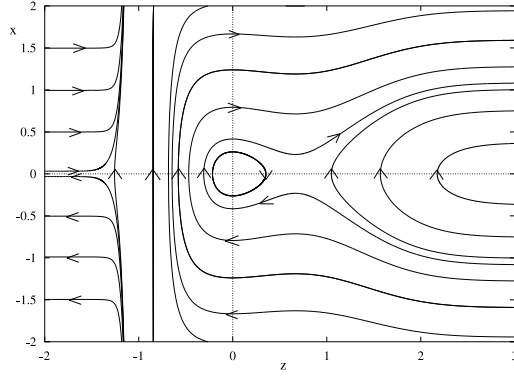


FIG. 16: The phase portrait (z, x) for case **A** $p = (\gamma_0 + \gamma_1 z + \gamma_2 z^2)\rho$ ($k = 1, \gamma_2 = 1$)

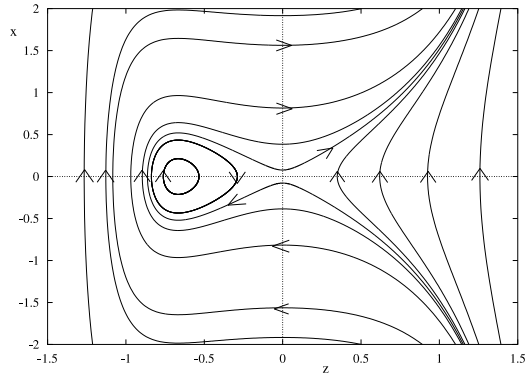


FIG. 17: The phase portrait (z, x) for case **A** $p = (\gamma_0 + \gamma_1 z + \gamma_2 z^2)\rho$ ($k = -1, \gamma_2 = -1$)

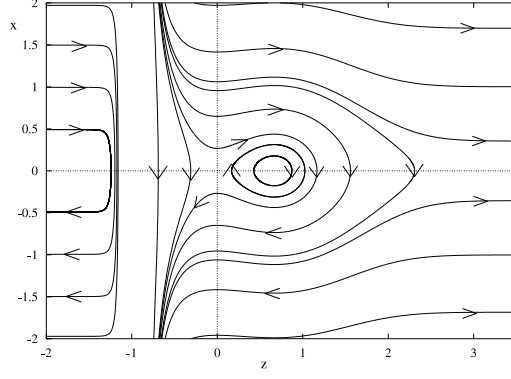


FIG. 18: The phase portrait (z, x) for case **A** $p = (\gamma_0 + \gamma_1 z + \gamma_2 z^2)\rho$ ($k = -1$, $\gamma_2 = 1$)

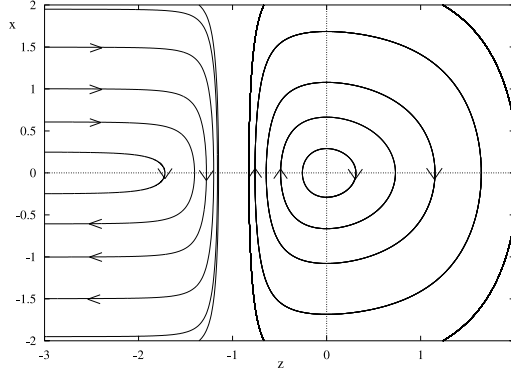


FIG. 19: The phase portrait (z, x) for case **B** $p = (\gamma_0 + \gamma_1 z)\rho$ ($k = 1$, $\gamma_0 = -1/3$, $\gamma_1 = -2/3$)

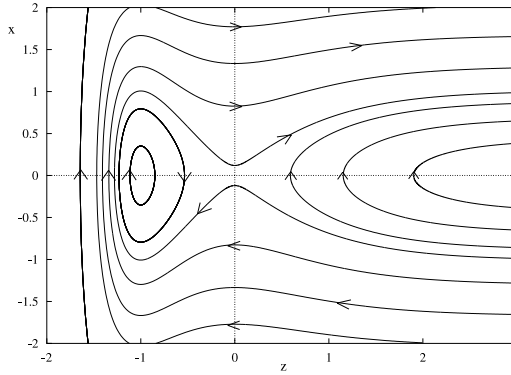


FIG. 20: The phase portrait (z, x) for case **B** $p = (\gamma_0 + \gamma_1 z)\rho$ ($k = 1$, $\gamma_0 = -1/3$, $\gamma_1 = 2/3$)

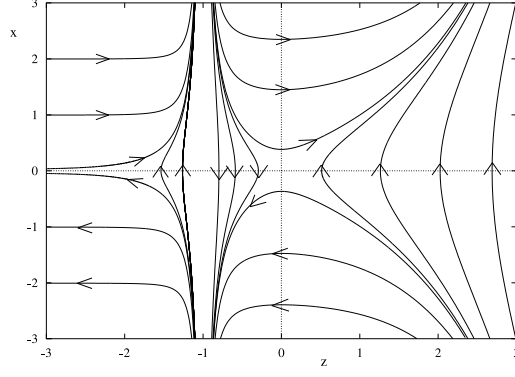


FIG. 21: The phase portrait (z, x) for case **B** $p = (\gamma_0 + \gamma_1 z)\rho$ ($k = -1$, $\gamma_0 = -1/3$, $\gamma_1 = -2/3$)

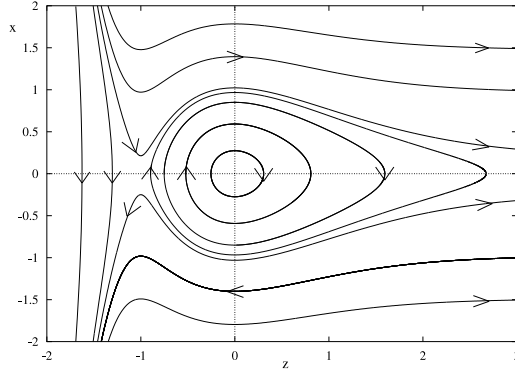


FIG. 22: The phase portrait (z, x) for case **B** $p = (\gamma_0 + \gamma_1 z)\rho$ ($k = -1$, $\gamma_0 = -1/3$, $\gamma_1 = 2/3$)

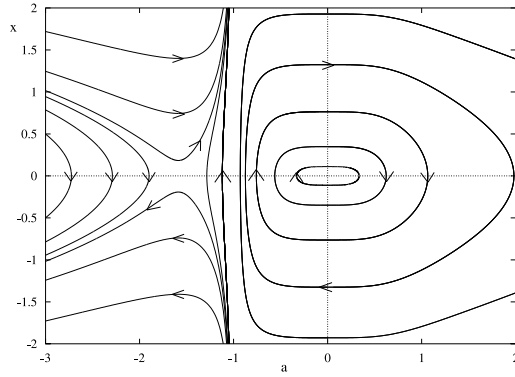


FIG. 23: The phase portrait (a, x) for case **C** $p = w_x \rho_x$, $k = 1$, $\rho_{0m}/\rho_{0x} = 1$, $w_x = 1$

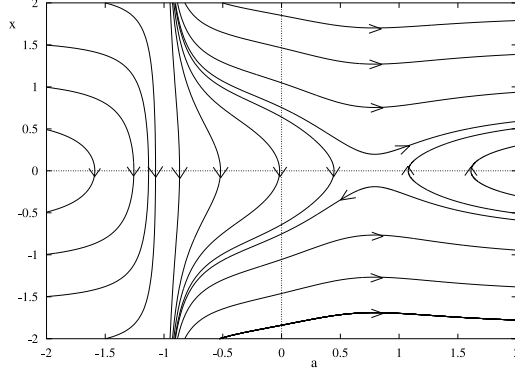


FIG. 24: The phase portrait (a, x) for case **C** $p = w_x \rho_x$, $k = 1$, $\rho_{0m}/\rho_{0x} = 1$, $w_x = -1$

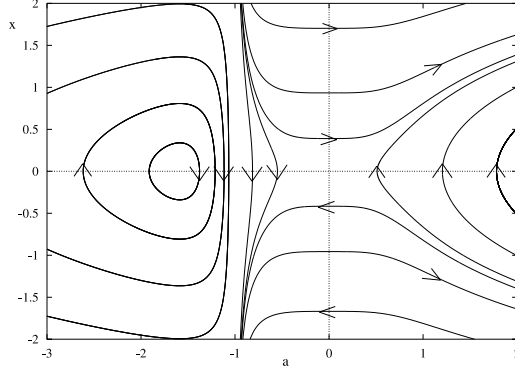


FIG. 25: The phase portrait (a, x) for case **C** $p = w_x \rho_x$, $k = -1$, $\rho_{0m}/\rho_{0x} = 1$, $w_x = 1$

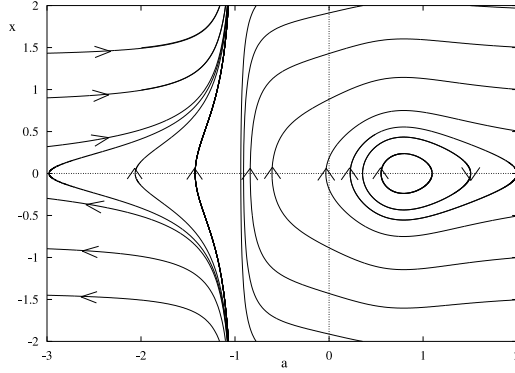


FIG. 26: The phase portrait (a, x) for case **C** $p = w_x \rho_x$, $k = -1$, $\rho_{0m}/\rho_{0x} = 1$, $w_x = -1$